A New Method for Designing Allpass Filters with Equiripple Group Delay Errors

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***Abstract*—Allpass filters have found many applications in signal processing areas. This paper describes an algorithm for the design of stable allpass digital filter with equiripple group delay errors. The problem is formulated as an iterative reweighted linear program (LP) problem. An algorithm is derived for solving such a problem. The design examples are given, which demonstrate that the proposed algorithm is very efficient and converges much faster than the existing ones for the design of allpass filters with equiripple constant group delay errors.**

***Index Terms*—allpass, equiripple, weighting function, iterative minimax**

I. Introduction

Digital allpass filters are a specific class of filters which only change the input signals' phase spectrum. In the recent years, allpass filters have been used in many digital signal processing applications [1]-[6] such as design and implementation of filter banks, halfband filters, notch filters, multirate filters, variable fractional delay filters and group-delay equalization, etc. Since an allpass filter has a unit magnitude across the whole frequency range $[0,\pi]$, the design of such a filter is actually an approximation of phase response. There are two widely used criteria for the approximation problem that are the minimax and the weighted least-squares [2], [4], [7]. Moreover, the minimax criterion minimizes the filter's maximum phase deviation from a desired one, which is better than the weighted least squares. Several algorithms for the minimax design are shown in the literature [8], [9]. All these algorithms can be used for designing allpass filters with equiripple phase errors. By incorporating an iterative minimax technique, the weighted least-squares method can also be applied to the equiripple phase and group delay errors design [10], [11].

It is noted [11] that a minimax allpass filter has an equiripple phase error and the maximum phase error is the smallest. However, the filter's group delay error is not equiripple. Moreover, the group delay deviation near the band edge is much larger than elsewhere in the frequency band of interest. It is shown that the minimax phase error design can not obtain an allpass filter with an equiripple group delay. In order to do that, some minimax or equiripple group delay error design methods had been proposed. In [5] an iterative reweighted least-squares group delay error method is used for designing allpass variable fractional delay filters with minimax group delay errors. A least pth group delay error method is also used to obtain allpass filters with equiripple group delay errors. As we all know that either the l_2 or the l_p norm of the group delay error is highly nonconvex, both methods are difficult to find true minimax group delay error filters.

Recently, a procedure was presented in [11] to design allpass digital filters with near equiripple group delay errors by iterative reweighted minimax which contains a stable constrain. However, the number of iterations required in [11] is large and the positive realness condition with a pole radius which is used to control the stability is too strict. The main objective in this paper is to propose a new approach to overcome these problems.

The main contributions are as follows. Instead of using the pole radius based constrain, the positive realness condition without a pole radius is used in our propose algorithm, which has been shown very efficient. In order to speed up the convergence, a new weighting function is proposed, which leads to an allpass filter with equiripple group delay errors. This method, referred to as an iterative reweighted minimax (IRWM) phase error method, is different from the method proposed in [11] though both of them use an iterative reweighted technique. In our method a new weighting function is utilized in the design and a less restrict stability constrain is proposed.

This paper is organized as follows. In Section II, the problem of designing allpass filter with equiripple phase error is introduced and the an existing minimax-based algorithm is presented which is to be compared with our proposed one. Our main contribution is found in Section III, where the optimal design problem is formulated with a less restrictive stability constraint and more efficient weighting function introduced. An algorithm is also proposed in this section. Section IV presents two design examples which demonstrate the superior convergence behavior and the effectiveness of the proposed algorithm. Some conclusions is given in Section V.

II. Preliminaries

An N-th order digital allpass filter can be described as

$$H_{ap}(z) \triangleq \frac{z^{-N}A(z^{-1})}{A(z)} \tag{1}$$

where $A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}$ and the filter coefficients vector is denoted by $\boldsymbol{a} \triangleq \begin{bmatrix} a_1 & \cdots & a_k & \cdots & a_N \end{bmatrix}^{\mathcal{T}}$, where $\mathcal{T}$ denotes the transpose operator.

By substituting z with $e^{j\omega}$ in the (1), we obtain the frequency response of the filter $H_{ap}(z)$ as follows:

$$H_{ap}(e^{j\omega}) \triangleq \frac{e^{-jN\omega}A(e^{-j\omega})}{A(e^{j\omega})} = e^{j\theta(\omega,\boldsymbol{a})} \tag{2}$$

where $\theta(\omega,\boldsymbol{a})$ is the filter's phase response and through some manipulators, it can be shown that:

$$\theta(\omega,\boldsymbol{a}) = -N\omega + 2\varphi(\omega,\boldsymbol{a}) \tag{3}$$

where

$$\varphi(\omega,\boldsymbol{a}) = \tan^{-1}\frac{\boldsymbol{s}^{\mathcal{T}}(\omega)\boldsymbol{a}}{1+\boldsymbol{c}^{\mathcal{T}}(\omega)\boldsymbol{a}} \tag{4}$$

with $\boldsymbol{s}(\omega)$ and $\boldsymbol{c}(\omega)$ given below:

$$\boldsymbol{s}(\omega) = \begin{bmatrix} \sin\omega & \cdots & \sin(k\omega) & \cdots & \sin(N\omega) \end{bmatrix}^{\mathcal{T}} \tag{5}$$

$$\boldsymbol{c}(\omega) = \begin{bmatrix} \cos\omega & \cdots & \cos(k\omega) & \cdots & \cos(N\omega \end{bmatrix}^{\mathcal{T}} \tag{6}$$

Assume that the desired phase frequency response is $\theta_d(\omega)$, which is defined on a dense grid of frequencies linearly distributed from $\omega = 0$ to $\omega = \pi$ to form a set of linear equations. Then the phase response error between $\theta_d(\omega)$ and the actual response is

$$E_\theta(\omega) = \theta(\omega,\boldsymbol{a}) - \theta_d(\omega) \tag{7}$$

Substituting (3) in (7), we have

$$\frac{E_\theta(\omega,\boldsymbol{a})}{2} = \varphi(\omega,\boldsymbol{a}) - \beta_d(\omega) \tag{8}$$

where $\beta_d(\omega) = \frac{N\omega+\theta_d(\omega)}{2}$.

By taking the tangent of both sides of (8) and using (4), we have

$$\tan\frac{E_\theta(\omega,\boldsymbol{a})}{2} = \frac{-\sin\beta_d(\omega) + \hat{\boldsymbol{s}}^{\mathcal{T}}(\omega)\boldsymbol{a}}{\cos\beta_d(\omega) + \hat{\boldsymbol{c}}^{\mathcal{T}}(\omega)\boldsymbol{a}} \tag{9}$$

where $\hat{\boldsymbol{s}}(\omega) = \boldsymbol{s}(\omega)\cos\beta_d(\omega) - \boldsymbol{c}(\omega)\sin\beta_d(\omega)$ and $\hat{\boldsymbol{c}}(\omega) = \boldsymbol{c}(\omega)\cos\beta_d(\omega) + \boldsymbol{s}(\omega)\sin\beta_d(\omega)$. By some simple algebra operations, the $\hat{\boldsymbol{s}}(\boldsymbol{\omega})$ and $\hat{\boldsymbol{c}}(\omega)$ can be described as:

$$\hat{\boldsymbol{s}}(\omega) = \begin{bmatrix} \sin[\omega - \beta_d(\omega)] \\ \sin[2\omega - \beta_d(\omega)] \\ \vdots \\ \sin[N\omega - \beta_d(\omega)] \end{bmatrix} \tag{10}$$

$$\hat{\boldsymbol{c}}(\omega) = \begin{bmatrix} \cos[\omega - \beta_d(\omega)] \\ \cos[2\omega - \beta_d(\omega)] \\ \vdots \\ \cos[N\omega - \beta_d(\omega)] \end{bmatrix} \tag{11}$$

Since the tangent function is monotonic increasing in $(-\frac{\pi}{2},\frac{\pi}{2})$ and $-\frac{\pi}{2} < E_\theta(\omega,\boldsymbol{a}) < \frac{\pi}{2}$, imposing an upper bound Δ on $|E_\theta(\omega,\boldsymbol{a})|$ is equivalent to impose an upper bound δ on $|\tan[\frac{E_\theta(\omega,\boldsymbol{a})}{2}]|$ with $\delta = |\tan(\frac{\Delta}{2})|$. Now the problem can be formed as a minimax phase error design of the allpass filter $H_{ap}(e^{j\omega})$ as follows:

$$\begin{aligned} &\min\nolimits_{\delta,\boldsymbol{a}} \quad \delta \\ &s.t. \quad \left|\frac{-\sin\beta_d(\omega)+\hat{S}(\omega_i)\boldsymbol{a}}{\cos\beta_d(\omega)+\hat{C}(\omega_i)\boldsymbol{a}}\right| \le \delta, \quad i = 1,2,\ldots,n_\Omega \end{aligned} \tag{12}$$

In order to solve this problem, the methods described in [7], [11] can be referred. For convenience the method proposed in [11] is used. The main procedure of this algorithm is given below:

Algorithm 1:

Phase 1: Let $\boldsymbol{a}(0) = \boldsymbol{0}$ and $k = 0$.

Phase 2: Solve the following problem for $\boldsymbol{a}(k+1)$:

$$\begin{aligned} &\boldsymbol{a}(k+1) = \arg\ \min\nolimits_{\delta,\boldsymbol{a}} \quad \delta \\ &s.t. \quad -\delta \le \frac{-\sin\beta_d(\omega)+\hat{S}(\omega_i)\boldsymbol{a}}{|\cos\beta_d(\omega)+\hat{C}(\omega_i)\boldsymbol{a}(k)|} \le \delta, \quad i = 1,2,\ldots,n_\Omega \end{aligned} \tag{13}$$

Phase 3: If

$$\frac{\max_{\omega_i}|E_\theta(\omega,\boldsymbol{a}(k+1))| - \max_{\omega_i}|E_\theta(\omega,\boldsymbol{a}(k))|}{\max_{\omega_i}|E_\theta(\omega,\boldsymbol{a}(k))|} > v \tag{14}$$

where $v > 0$ is a relative tolerance of the maximun phase error, let $k = k+1$ and go back to **Phase** 2. Otherwise, terminate the algorithm.

Comments:

- Equation (13) is an LP problem which can be solved by many toolbox, Such as Matlab optimization toolbox and CVX [12].
- Algorithm 1 is used to obtain an allpass digital filter with equiripple phase error, which yields a fast convergence.
- Note that the allpass digital filter is an IIR filter, the stability should be considered in the process of optimal design. In the following section a simple constraint will be proposed which can guarantee the stability efficient.

III. The proposed algorithm

This section shall present a constrain to guarantee the filter stability and weighting function design which can get an equiripple group delay error. Moreover the proposed algorithm is also given.

A. Stability

The stability of $A(z)$ is guaranteed if

$$\mathrm{Re}[A(e^{j\omega})] > 0, \quad \text{for} \quad \omega \in [0,\pi] \tag{15}$$

In practical operation the $\mathrm{Re}[A(e^{j\omega})] \ge \xi$ will be used instead of equation (15), Where ξ is a very small positive value to ensure a reasonable stability margin [13]. The discrete version of (15) is implemented, i.e., $\mathrm{Re}[A(e^{j\omega})] \ge \xi > 0$, for $\omega \in S_{\Omega_s} = \{\omega_i, i = 1,2,\ldots,n_{\Omega_s}\}$. Due to this, a set of linear constraints on $\boldsymbol{a}$ have been formed:

$$\boldsymbol{A}_{\Omega_s}\boldsymbol{a} \le (1-\xi)\boldsymbol{e}_{\Omega_s} \tag{16}$$

where

$$\boldsymbol{A}_{\Omega_s} = -\begin{bmatrix} \cos\omega_1 & \cos 2\omega_1 & \cdots & \cos N\omega_1 \\ \cos\omega_2 & \cos 2\omega_2 & \cdots & \cos N\omega_2 \\ \vdots & \vdots & \vdots & \vdots \\ \cos\omega_{n_{\Omega_s}} & \cos 2\omega_{n_{\Omega_s}} & \cdots & \cos N\omega_{n_{\Omega_s}} \end{bmatrix}$$

$$\boldsymbol{e}_{n_{\Omega_s}} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^{\mathcal{T}} \in \Re^{n_{\Omega_s}\times 1}$$

It should be pointed that the condition in (15) is *sufficient* to ensure the stability. However the constraints are linear and can be handled easily. In fact the constraints in (15) are often restrictive which can guarantee the stability simple. Fortunately, the discrete version of the condition, namely (16), offers some flexibility in controlling filter's stability. On one hand, with a proper dense S_{Ω_s}, the constraints in (16) can be made arbitrary close to (15) so as to guarantee the stability. On the other hand, from our experience a sparse S_{Ω_s} can also guarantee the stability and have less restrictive. In our following simulation the constraints in (16) with an n_{Ω_s} lesser than the order is also sufficient to yield a stable design.

B. Weighting function $W(\omega)$

To guarantee an allpass filter design with equiripple group delay errors, a weighting function is proposed in [11] which is imposed on the constrains described as (13). However the algorithm's convergence in this weighting function is slower. In order to find a better weighting function to accelerate the convergence. A new weighting function proposed in [14] is used in our method. We think the weighting function shown in [11] using the information about the envelope is seldom, so the convergence is slow. In fact the weighting function proposed in [14] by Lim can be used to solve this problem. In this correspondence, Lim's weighting function is used directly.

The algorithm for the weighting function design is given as follows:

- Given an initial weight function $W_0(\omega)$ for $\omega \in S_\Omega$ which can set to be $[1, 1, \cdots, 1]$.
- A new $\boldsymbol{a}(k)$ will produce a new $E_{\theta_k}(\omega)$. Moreover the group delay error can be describe as $E_{g_k}(\omega) = \frac{\partial E_{\theta_k}(\omega)}{\partial\omega}$. The envelop of $|E_{g_k}(\omega)|$ is $\Theta_k(\omega)$. Define the ith extremal group delay errors at the kth iteration as the errors where $\Theta_k(\omega_{J(i)}) > \Theta_k(\omega_{J(i)\pm 1})$. Band edges are considered as extremal errors. The extremal group delay errors should be labeled consecutively so that $\omega_{J(i+1)} > \omega_{J(i)}$ for $i = 1, 2, \cdots$. We define the ith extremal point of the kth iteration as
$$V_k(i) = \Theta_k(\omega_{J(i)}) \tag{17}$$
It is noted that the $V_k(i)$ too small or even 0 will yields the weighting to zero in every iteration. To avoid this, for any nonband edge $V_k(i)$ less than 0.1 of $\min(V_k(i-1), V_k(i+1))$, we shall arbitrarily define $V_k(i)$ to be 0.1 of the $\min(V_k(i-1), V_k(i+1))$.
- The new envelop function, $\alpha_k(\omega)$, is formed by joining together all the extremal points of the same frequency band of interest using straight lines. Extremal points of different frequency band are not jointed together. For $\omega_{J(i)} < \omega < \omega_{J(i+1)}$, $\alpha_k(\omega)$ is given by
$$\alpha_k(\omega) = \frac{\omega - \omega_{J(i)}}{\omega_{J(i+1)} - \omega_{J(i)}} V_k(i+1) + \frac{\omega_{J(i+1)} - \omega}{\omega_{J(i+1)} - \omega_{J(i)}} V_k(i) \tag{18}$$
where $\omega_{J(i+1)}$ and $\omega_{J(i)}$ are of the same frequency band.
- Define a function $\Xi_k(\omega_i) = (\tau\alpha_k(\omega_i))^{1.5}$, for $i = 1, 2, \ldots, n_\Omega$. where τ is the average value of $\frac{1}{\alpha_k(\omega_i)}$, for $i = 1, 2, \ldots, n_\Omega$.
- Reform the weighting function as $W_{k+1}(\omega) = W_k(\omega)\Xi_k(\omega)$.

Based on the discussions above, an *iterative reweighted minimax* method is proposed to obtain an allpass digital filter with equiripple group delay error. The main steps of the proposed algorithm are given below:

Algorithm 2:

- *Step I*: Set the initial values $\boldsymbol{a}(0), W_0(\omega)$ which can get from the result of algorithm 1 and set $k = 0$.
- *Step II*: Solve the following problem for $\boldsymbol{a}(k+1)$:
$$\begin{aligned} &\boldsymbol{a}(k+1) = \arg\min_{\delta,\boldsymbol{a}} \quad \delta \\ s.t.\quad &-\delta \le \tfrac{W_k(\omega_i)(-\sin\beta_d(\omega) + \hat{S}(\omega_i)\boldsymbol{a})}{|\cos\beta_d(\omega) + \hat{C}(\omega_i)\boldsymbol{a}(k)|} \le \delta, \quad i = 1, 2, \ldots, n_\Omega \\ &\boldsymbol{A}_{\Omega_s}\boldsymbol{a} \le (1-\xi)\boldsymbol{e}_{\Omega_s} \end{aligned} \tag{19}$$
- *Step III*: Through the $\boldsymbol{a}(k+1)$ form the new weighting function $W_{k+1}(\omega)$
- *Step VI*: Let ρ_{ave} be the average group delay errors' ripple magnitude. The algorithm will be terminated once (20) is satisfied:
$$\rho_{max} \le (1+\gamma)\rho_{ave} \tag{20}$$
where ρ_{max} is the maximum ripple magnitude in the group delay errors and γ is a small positive tolerance. Otherwise, let $k = k+1$ and go back to *Step VI*.

Comments:

- The initial selection in the algorithm 2 can reduce the iteration about 1 or 2 for our experience. So this selection only a custom will not affect the convergence speed severely.
- A pre-specified number of iterations can also be used to terminate algorithm , the final available solution is also good.

IV. Design Numerical Examples

In this section, two examples are presented to demonstrate the performance of **Algorithm 1** given in [11]and the proposed **Algorithm 2** in this paper. For comparison purpose, the examples used in [11] are adopted in this paper.

Example 1: Design an 8-th order allpass filter with a designed linear phase response $\theta_d(\omega) = -0.70615\omega$ defined on $\Omega = [0, 0.8\pi]$. The n_Ω and n_{Ω_s} are set to 401 and 6, respectively. Moreover the algorithm will be terminated, when

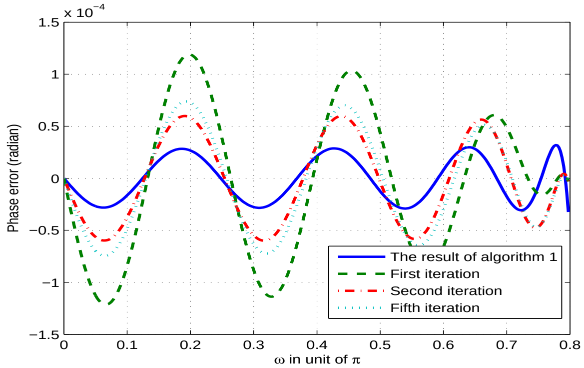

Fig. 1. Phase error response of the allpass filters

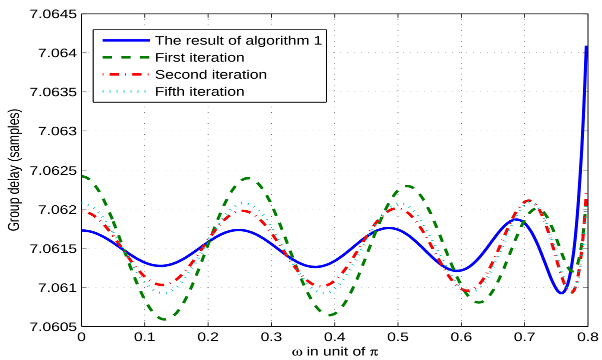

Fig. 2. Group delay response of the filters

the maximum phase error smaller than 3.274×10^{-5} which is proposed in [11].

Algorithm 1 converges very fast. It takes 1 iteration, resulting in a filter with a maximum phase and group delay errors of 3.251×10^{-5} and 2.591×10^{-3}, respectively, which are smaller and faster than that of [11]. Moreover the poles of the filter obtained are given in Table I. The largest magnitude of the poles is 0.9785. The filter's phase error and group delay responses on ω are shown in Figs. 1 and 2, respectively. It is noted that the phase error is equiripple, while the group delay error is not. It can be seen from Figs. 1 and 2 that a large group delay error is near the high frequency edge than elsewhere. **Algorithm 2** can solve this problem successfully and presented in *Example 2*.

TABLE I
POLES OF THE TRANSFER FUNCTION IN EXAMPLE 1

Poles
-0.97848456
0.24207910
$-0.24763441 \pm 0.17880626j$
$-0.02002978 \pm 0.26115500j$
$0.16765515 \pm 0.18030637j$

Example 2: In this example we use the same specifications as Example 1 and the result in Example 1 as the initial value. Running the algorithm 2, we get a better result than [11] after 2 iterations. In our algorithm the maximum phase and group delay errors are 5.996×10^{-5}, 7.113×10^{-4}.

In this example the poles of the filter obtained are given in Table II. The largest magnitude of the poles is 0.9795. Additionally the phase error responses and the group delay responses in every iterations are shown in Figs. 1 and 2.

TABLE II
POLES OF THE TRANSFER FUNCTION IN EXAMPLE 2

Poles
-0.97952308
0.26007895
$-0.26152806 \pm 0.18446780j$
$-0.02510860 \pm 0.27714599j$
$0.17804754 \pm 0.19466024j$

Comments:

- As seen from the simulations, our proposed algorithm converges much faster **Algorithm 1**. The resultant filters yield the maximum group delay error much smaller, while the phase error is larger. This is due to the fact that the weighting function lets the group delay error evenly distribution in all of the interesting frequency interval.
- In our simulations, **Algorithm 2** in this example converges after 5 iterations. It was also observed that after 2 iterations the change is in the same order of magnitudes.

V. CONCLUSIONS

A new approach has been proposed in this paper for the minimax phase error design of allpass filters. In order to obtain equiripple group delay errors, an iterative reweighted minimax method has been proposed. The new method converges faster than the method presented in [11] applied in the design of allpass filter with equiripple group delays.

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